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Families of solutions of the nested Bethe ansatz for the A_2 spin chain

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Abstract. The full set of polynomial solutions of the nested Bethe ansatz is constructed for the case of the A_2 rational spin chain. The structure and properties of these associated solutions are more various than the case of the usual *XXX* (A_1) spin chain but their role is similar.

1. Introduction

In our previous paper [1] we considered the famous Baxter *TQ* equations [5] for the simplest cases of the *XXX* and *XXZ* spin chains. In particular, we showed that for each solution $Q(\lambda)$ of the Bethe equations there exists an associated solution $P(\lambda)$ that corresponds to the same eigenvalue $T(\lambda)$ of the transfer matrix. The associated solution does not define a 'physical' Bethe state; however, it is found to be useful in its own right.

The polynomials $Q(\lambda)$ and $P(\lambda)$ form a full set of solutions of the *TQ* equation

$$T(\lambda)Q(\lambda) = (\lambda - \mathrm{i}/2)^N Q(\lambda + \mathrm{i}) + (\lambda + \mathrm{i}/2)^N Q(\lambda - \mathrm{i}) \tag{1}$$

which may be considered to be a second-order finite-difference equation with respect to $Q(\lambda)$. As for second-order differential equations we can express the coefficients of (1) via its solutions Q and P as

$$P(\lambda + \mathrm{i}/2)Q(\lambda - \mathrm{i}/2) - P(\lambda - \mathrm{i}/2)Q(\lambda + \mathrm{i}/2) = \lambda^N \tag{2}$$

$$T(\lambda) = P(\lambda + \mathrm{i})Q(\lambda - \mathrm{i}) - P(\lambda - \mathrm{i})Q(\lambda + \mathrm{i}). \tag{3}$$

It is remarkable that the set of polynomial solutions of (2) reproduce the spectrum $T(\lambda)$ via (3).

The construction above corresponds to the case when the fundamental set of quantum operators belongs to the algebra A_1 (and its deformations). In this paper we take the first step in the generalization of our approach to the algebras A_n. For the sake of simplicity we limit ourselves to the isotropic A_2 spin chain, setting aside its deformations for future publications. We show that each solution of the nested Bethe ansatz equations is associated with five additional solutions that correspond to the same eigenvalue of the transfer matrix. Also, we show that the third-order finite-difference equation, which is an analogue of Baxter's equation for the case of A_2, has a full set of polynomial solutions Q, P and R. The corresponding 'Wronskian' has the form

$$\begin{vmatrix} Q(\lambda - \mathrm{i}) & Q(\lambda) & Q(\lambda + \mathrm{i}) \\ P(\lambda - \mathrm{i}) & P(\lambda) & P(\lambda + \mathrm{i}) \\ R(\lambda - \mathrm{i}) & R(\lambda) & R(\lambda + \mathrm{i}) \end{vmatrix} = \lambda^N. \tag{4}$$

The polynomial solutions of this equation are the components of the full spectrum of the A_2 transfer matrix. For example, the eigenvalues of the transfer matrices corresponding to the two fundamental representations are given by

$$\begin{vmatrix} Q(\lambda - 3\mathrm{i}/2) & Q(\lambda \pm \mathrm{i}/2) & Q(\lambda + 3\mathrm{i}/2) \\ P(\lambda - 3\mathrm{i}/2) & P(\lambda \pm \mathrm{i}/2) & P(\lambda + 3\mathrm{i}/2) \\ R(\lambda - 3\mathrm{i}/2) & R(\lambda \pm \mathrm{i}/2) & R(\lambda + 3\mathrm{i}/2) \end{vmatrix} = T^{\pm}(\lambda). \tag{5}$$

These equations replace (2) and (3) for the case of A_2.

2. Various formulations of the nested Bethe ansatz

The exact formulation of the model can be found in for example [2]. Diagonalization of the transfer matrix and corresponding Hamiltonian has been accomplished with the help of the so-called nested Bethe ansatz [3], which can be constructed in the framework of QISM (see e.g. [4]).

Let us recall the general setup of the nested Bethe ansatz equations for the case of an A_2 spin chain Take N to be the length of the chain and introduce non-negative integers n_1 and n_2 that satisfy

$$n_1 \leqslant N/3 \qquad n_2 \leqslant 2N/3 \qquad 2n_1 \leqslant n_2. \tag{6}$$

The corresponding Bethe state is defined by $n_1 + n_2$ parameters, which we denote by

$$\lambda_j^{(1)} (j = 1, 2, \ldots, n_1) \qquad \lambda_k^{(2)} (k = 1, 2, \ldots, n_2). \tag{7}$$

The equations for $\lambda_j^{(1)}$ and $\lambda_k^{(2)}$ are

$$\begin{gathered} \prod_{j'=1}^{n_1} \frac{\lambda_j^{(1)} - \lambda_{j'}^{(1)} + \mathrm{i}}{\lambda_j^{(1)} - \lambda_{j'}^{(1)} - \mathrm{i}} \times \prod_{k'=1}^{n_2} \frac{\lambda_j^{(1)} - \lambda_{k'}^{(2)} - \frac{\mathrm{i}}{2}}{\lambda_j^{(1)} - \lambda_{k'}^{(2)} + \frac{\mathrm{i}}{2}} = -1 \qquad (j = 1, 2, \ldots, n_1) \\ \prod_{j'=1}^{n_1} \frac{\lambda_k^{(2)} - \lambda_{j'}^{(1)} - \frac{\mathrm{i}}{2}}{\lambda_k^{(2)} - \lambda_{j'}^{(1)} + \frac{\mathrm{i}}{2}} \times \prod_{k'=1}^{n_2} \frac{\lambda_k^{(2)} - \lambda_{k'}^{(2)} + \mathrm{i}}{\lambda_k^{(2)} - \lambda_{k'}^{(2)} - \mathrm{i}} = -\left(\frac{\lambda_k^{(2)} + \frac{\mathrm{i}}{2}}{\lambda_k^{(2)} - \frac{\mathrm{i}}{2}} \right)^N \qquad (k = 1, 2, \ldots, n_2). \end{gathered} \tag{8}$$

Let us define a pair of polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$ of degrees n_1 and n_2 respectively, by

$$Q_1(\lambda) = \prod_{j'=1}^{n_1} (\lambda - \lambda_{j'}^{(1)}) \qquad Q_2(\lambda) = \prod_{k'=1}^{n_2} (\lambda - \lambda_{k'}^{(2)}). \tag{9}$$

Making use of these polynomials we can rewrite (8) as

$$\begin{gathered} Q_1(\lambda_j^{(1)} + \mathrm{i}) Q_2\left(\lambda_j^{(1)} - \frac{\mathrm{i}}{2}\right) + Q_1(\lambda_j^{(1)} - \mathrm{i}) Q_2\left(\lambda_j^{(1)} + \frac{\mathrm{i}}{2}\right) = 0 \qquad (j = 1, 2, \ldots, n_1) \\ \left(\lambda_k^{(2)} + \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda_k^{(2)} + \frac{\mathrm{i}}{2}\right) Q_2(\lambda_k^{(2)} - \mathrm{i}) + \left(\lambda_k^{(2)} - \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda_k^{(2)} - \frac{\mathrm{i}}{2}\right) \\ \times Q_2(\lambda_k^{(2)} + \mathrm{i}) = 0 \qquad (k = 1, 2, \ldots, n_2). \end{gathered} \tag{10}$$

If all the roots of $Q_1(\lambda)$ and $Q_2(\lambda)$ are simple then (10) implies

$$Q_2\left(\lambda + \frac{\mathrm{i}}{2}\right) Q_1(\lambda - \mathrm{i}) + Q_2\left(\lambda - \frac{\mathrm{i}}{2}\right) Q_1(\lambda + \mathrm{i}) = T_2(\lambda) Q_1(\lambda) \tag{11}$$

$$\left(\lambda + \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda + \frac{\mathrm{i}}{2}\right) Q_2(\lambda - \mathrm{i}) + \left(\lambda - \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda - \frac{\mathrm{i}}{2}\right) Q_2(\lambda + \mathrm{i}) = T_1(\lambda) Q_2(\lambda) \tag{12}$$

where $T_1(\lambda)$ and $T_2(\lambda)$ are new polynomials, the meaning of which will be clarified later.

The eigenvalues $T(\lambda)$ of the transfer matrix enter the game via the following construction. Shifting the argument in (11) by $\pm\frac{\mathrm{i}}{2}$ and combining the result with (12), we obtain

$$\left\{T_1(\lambda) + \left(\lambda \pm \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda \mp \frac{3\mathrm{i}}{2}\right)\right\} Q_2(\lambda)$$
$$= \left\{\left(\lambda \pm \frac{\mathrm{i}}{2}\right)^N T_2\left(\lambda \mp \frac{\mathrm{i}}{2}\right) + \left(\lambda \mp \frac{\mathrm{i}}{2}\right)^N Q_2(\lambda \pm \mathrm{i})\right\} Q_1\left(\lambda \mp \frac{\mathrm{i}}{2}\right). \tag{13}$$

Suppose now that $Q_2(\lambda)$ and $Q_1(\lambda \pm \frac{\mathrm{i}}{2})$ are mutually simple, i.e. have no common roots. Then (13) implies the following important formulae:

$$\begin{gathered} T_1(\lambda) + \left(\lambda \pm \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda \mp \frac{3\mathrm{i}}{2}\right) = T^{\pm}(\lambda) Q_1\left(\lambda \mp \frac{\mathrm{i}}{2}\right) \\ \left(\lambda \pm \frac{\mathrm{i}}{2}\right)^N T_2\left(\lambda \mp \frac{\mathrm{i}}{2}\right) + \left(\lambda \mp \frac{\mathrm{i}}{2}\right)^N Q_2(\lambda \pm \mathrm{i})\} = T^{\pm}(\lambda) Q_2(\lambda) \end{gathered} \tag{14}$$

where $T^{\pm}(\lambda)$ are polynomials of degree N, corresponding to eigenvalues of the transfer matrices associated with the adjoint fundamental representations of the A_2 auxiliary space.

We can eliminate $T_1(\lambda)$ and $T_2(\lambda)$ by using (14), to obtain

$$\left(\lambda + \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda - \frac{3\mathrm{i}}{2}\right) - T^{+}(\lambda) Q_1\left(\lambda - \frac{\mathrm{i}}{2}\right)$$
$$+ T^{-}(\lambda) Q_1\left(\lambda + \frac{\mathrm{i}}{2}\right) - \left(\lambda - \frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda + \frac{3\mathrm{i}}{2}\right) = 0 \tag{15}$$

$$\lambda^N (\lambda + \mathrm{i})^N Q_2\left(\lambda - \frac{3\mathrm{i}}{2}\right) - (\lambda + \mathrm{i})^N T^{-}\left(\lambda - \frac{\mathrm{i}}{2}\right) Q_2\left(\lambda - \frac{\mathrm{i}}{2}\right)$$
$$+ (\lambda - \mathrm{i})^N T^{+}\left(\lambda + \frac{\mathrm{i}}{2}\right) Q_2\left(\lambda + \frac{\mathrm{i}}{2}\right) - \lambda^N (\lambda - \mathrm{i})^N Q_2\left(\lambda + \frac{3\mathrm{i}}{2}\right) = 0. \tag{16}$$

These equations, which we shall encounter again later on, can be solved for $T^{\pm}(\lambda)$

$$T^{\pm}\left(\lambda \pm \frac{\mathrm{i}}{2}\right) Q_1(\lambda) Q_2\left(\lambda \pm \frac{\mathrm{i}}{2}\right) = \lambda^N Q_1(\lambda) Q_2\left(\lambda \pm \frac{3\mathrm{i}}{2}\right)$$
$$+ (\lambda \pm \mathrm{i})^N \left\{Q_1(\lambda + \mathrm{i}) Q_2\left(\lambda - \frac{\mathrm{i}}{2}\right) + Q_1(\lambda - \mathrm{i}) Q_2\left(\lambda + \frac{\mathrm{i}}{2}\right)\right\}. \tag{17}$$

In contrast with (11), (12), these equations are homogeneous with respect to Q_1 and Q_2 and do not contain auxiliary polynomials T_1 and T_2. Also, (17) is equivalent to (11), (12) since the rhs of (17) divides $Q_1(\lambda)$ and $Q_2(\lambda \pm \frac{\mathrm{i}}{2}))$ according to (11), (12).

3. Associated solutions of the 'nested' Bethe ansatz

Let us denote the system (11), (12) by

$$\{Q_1, Q_2; T_1, T_2\}. \tag{18}$$

The two components of (18) are each a kind of *TQ* equation [5] for some inhomogeneous *XXX* spin chain. For example (11) may be considered as the *TQ* equation for a chain of length n_2 with inhomogeneities defined by the roots of $Q_2(\lambda)$. According to [1] there exists a polynomial $P_1(\lambda)$ of degree $n_2 - n_1 + 1$, which together with $Q_2(\lambda)$ satisfies

$$Q_2\left(\lambda - \frac{\mathrm{i}}{2}\right) P_1(\lambda + \mathrm{i}) + Q_2\left(\lambda + \frac{\mathrm{i}}{2}\right) P_1(\lambda - \mathrm{i}) = T_2(\lambda) P_1(\lambda). \tag{19}$$

We also have that Q_2 and T_2, which play the role of coefficients in (11), may be expressed in terms of two independent solutions Q_1 and P_1 as

$$\begin{aligned} Q_2(\lambda) &= P_1\left(\lambda+\frac{\mathrm{i}}{2}\right)Q_1\left(\lambda-\frac{\mathrm{i}}{2}\right) - P_1\left(\lambda-\frac{\mathrm{i}}{2}\right)Q_1\left(\lambda+\frac{\mathrm{i}}{2}\right) \\ T_2(\lambda) &= P_1(\lambda+\mathrm{i})Q_1(\lambda-\mathrm{i}) - P_1(\lambda-\mathrm{i})Q_1(\lambda+\mathrm{i}). \end{aligned} \tag{20}$$

Equation (12) may also be considered as a TQ equation but for a spin chain of length $N+n_1$. Now polynomial $\lambda^N Q_1(\lambda)$ serves as an inhomogeneity. Again, according to [1] the second solution $P_2(\lambda)$ is a polynomial of degree $N+n_1-n_2+1$ and we have

$$\left(\lambda-\frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda-\frac{\mathrm{i}}{2}\right)P_2(\lambda+\mathrm{i}) + \left(\lambda+\frac{\mathrm{i}}{2}\right)^N Q_1\left(\lambda+\frac{\mathrm{i}}{2}\right)P_2(\lambda-\mathrm{i}) = T_1(\lambda)P_2(\lambda). \tag{21}$$

A construction similar to (20) yields

$$\begin{aligned} \lambda^N Q_1(\lambda) &= P_2\left(\lambda+\frac{\mathrm{i}}{2}\right)Q_2\left(\lambda-\frac{\mathrm{i}}{2}\right) - P_2\left(\lambda-\frac{\mathrm{i}}{2}\right)Q_2\left(\lambda+\frac{\mathrm{i}}{2}\right) \\ T_1(\lambda) &= P_2(\lambda+\mathrm{i})Q_2(\lambda-\mathrm{i}) - P_2(\lambda-\mathrm{i})Q_2(\lambda+\mathrm{i}). \end{aligned} \tag{22}$$

Combining the first equation of (20) with that of (22) and excluding Q_2 we obtain the factorized equation

$$\begin{aligned} &Q_1(\lambda)\left\{\lambda^N + P_2\left(\lambda-\frac{\mathrm{i}}{2}\right)P_1(\lambda+\mathrm{i}) + P_2\left(\lambda+\frac{\mathrm{i}}{2}\right)P_1(\lambda-\mathrm{i})\right\} \\ &\qquad = P_1(\lambda)\left\{P_2\left(\lambda-\frac{\mathrm{i}}{2}\right)Q_1(\lambda+\mathrm{i}) + P_2\left(\lambda+\frac{\mathrm{i}}{2}\right)Q_1(\lambda-\mathrm{i})\right\}. \end{aligned} \tag{23}$$

Suppose $Q_1(\lambda)$ and $P_1(\lambda)$ are mutually simple (this is equivalent to the mutual simplicity of $Q_2(\lambda)$ and $Q_1(\lambda \pm \frac{\mathrm{i}}{2})$). Then there exists a polynomial $\tilde{T}_2(\lambda)$ satisfying

$$\begin{aligned} &P_2\left(\lambda+\frac{\mathrm{i}}{2}\right)Q_1(\lambda-\mathrm{i}) + P_2\left(\lambda-\frac{\mathrm{i}}{2}\right)Q_1(\lambda+\mathrm{i}) = \tilde{T}_2(\lambda)Q_1(\lambda) \\ &P_2\left(\lambda+\frac{\mathrm{i}}{2}\right)P_1(\lambda-\mathrm{i}) + P_2\left(\lambda-\frac{\mathrm{i}}{2}\right)P_1(\lambda+\mathrm{i}) + \lambda^N = \tilde{T}_2(\lambda)P_1(\lambda). \end{aligned} \tag{24}$$

Remarkably (21) and the first equation of (24) form a new pair of equations for the nested Bethe ansatz, which in our notation can be written as $\{Q_1, P_2; T_1, \tilde{T}_2\}$. Note that according to the first equation of system (14), this pair corresponds to the same eigenvalues of the transfer matrices $T^{\pm}(\lambda)$ as in the case of $\{Q_1, Q_2; T_1, T_2\}$.

Repeating the above procedure, but this time excluding Q_1, we arrive at

$$\begin{aligned} &\left(\lambda+\frac{\mathrm{i}}{2}\right)^N P_1\left(\lambda+\frac{\mathrm{i}}{2}\right)Q_2(\lambda-\mathrm{i}) + \left(\lambda-\frac{\mathrm{i}}{2}\right)^N P_1\left(\lambda-\frac{\mathrm{i}}{2}\right)Q_2(\lambda+\mathrm{i}) = \tilde{T}_1(\lambda)Q_2(\lambda) \\ &\left(\lambda+\frac{\mathrm{i}}{2}\right)^N P_1\left(\lambda+\frac{\mathrm{i}}{2}\right)P_2(\lambda-\mathrm{i}) + \left(\lambda-\frac{\mathrm{i}}{2}\right)^N P_1\left(\lambda-\frac{\mathrm{i}}{2}\right)P_2(\lambda+\mathrm{i}) \\ &\qquad + \left(\lambda-\frac{\mathrm{i}}{2}\right)^N \left(\lambda+\frac{\mathrm{i}}{2}\right)^N = \tilde{T}_1(\lambda)P_2(\lambda) \end{aligned} \tag{25}$$

which is the system $\{P_1, Q_2; \tilde{T}_1, T_2\}$. We may summarize this section in the following proposition.

Proposition 1. *(On the association between solutions of the nested Bethe ansatz equations.)*

If we have solution $\{Q_1, Q_2; T_1, T_2\}$ *of the Bethe equations (11), (12) and the degrees of the polynomials are* $(n_1, n_2; N+n_1, n_2)$ *respectively, then there exists a pair of associated solutions* $\{Q_1, P_2; T_1, \tilde{T}_2\}$ *and* $\{P_1, Q_2; \tilde{T}_1, T_2\}$ *for which the degrees are* $(n_1, N+n_1-n_2+1; N+n_1, N+n_1-n_2+1)$ *and* $(n_2-n_1+1, n_2; N+n_2-n_1+1, n_2)$.

4. The family of solutions of the nested Bethe ansatz equations

Each of the two associated solutions $\{Q_1, P_2; T_1, \tilde{T}_2\}$ and $\{P_1, Q_2; \tilde{T}_1, T_2\}$ can be considered the result of two operations $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively acting on the initial solution $\{Q_1, Q_2; T_1, T_2\}$, i.e.

$$\begin{aligned} \mathcal{F}_1\{Q_1, Q_2; T_1, T_2\} &= \{P_1, Q_2; \tilde{T}_1, T_2\} \\ \mathcal{F}_2\{Q_1, Q_2; T_1, T_2\} &= \{Q_1, P_2; T_1, \tilde{T}_2\}. \end{aligned} \tag{26}$$

One may obtain the impression that there may possibly exist an infinite set of associated solutions. However, below we find that $\mathcal{F}_1$ and $\mathcal{F}_2$ form a finite group, thus guaranteeing a finite number of associated solutions.

Firstly, let us remark that $\mathcal{F}_1$ and $\mathcal{F}_2$ are involutions

$$\mathcal{F}_1^2 = \mathcal{F}_2^2 = I. \tag{27}$$

Next, we have that the products $\mathcal{F}_2\mathcal{F}_1, \mathcal{F}_1\mathcal{F}_2\mathcal{F}_1, \ldots$ etc, form a finite set because it will be shown that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the Artin relation

$$\mathcal{F}_1\mathcal{F}_2\mathcal{F}_1 = \mathcal{F}_2\mathcal{F}_1\mathcal{F}_2. \tag{28}$$

This relation can be diagrammatically represented as

$$\begin{tikzcd} & \{Q_1, Q_2; T_1, T_2\} \arrow[dl, "\mathcal{F}_1"'] \arrow[dr, Rightarrow, "\mathcal{F}_2"] & \\ \{P_1, Q_2; \tilde{T}_1, T_2\} \arrow[d, Rightarrow, "\mathcal{F}_2"] & & \{Q_1, P_2; T_1, \tilde{T}_2\} \arrow[d, "\mathcal{F}_1"'] \\ \{P_1, R_2; \tilde{T}_1, T_2'\} \arrow[dr, "\mathcal{F}_1"'] & & \{R_1, P_2; T_1', \tilde{T}_2\} \arrow[dl, Rightarrow, "\mathcal{F}_2"] \\ & \{R_1, R_2; T_1', T_2'\}\,. & \end{tikzcd} \tag{29}$$

To prove this statement let us recall that the $T^{\pm}(\lambda)$ defined in (14) are invariant under the action of $\mathcal{F}_1$ and $\mathcal{F}_2$. Each of these operations does not change one of the two pairs $Q_{\mathrm{i}}, T_{\mathrm{i}}$ and due to (14) it is sufficient for the conservation of $T^{\pm}(\lambda)$.

Now let us consider equations (15), (16) from the first section. These equations may be considered as linear homogeneous finite-difference equations of the third order for polynomials Q_1 and Q_2. The invariants $T^{\pm}(\lambda)$ play the role of coefficients. Each has three linearly independent solutions Q_1, P_1, R_1, and Q_2, P_2, R_2, respectively. If (28) is not valid, i.e. the chain of solutions (29) is longer, then we should obtain more than three solutions to each of the equations (15), (16), which is impossible.

5. Concluding remarks

In our previous paper [1] we considered two fundamental polynomial solutions to Baxter's *TQ* equation. These solutions may be considered as fundamental objects of the integrable A_1 spin chain model. They give rise to all possible fusion relations for the transfer matrices corresponding to different spins in the auxiliary space and the transfer matrices themselves can be expressed in terms of these polynomial solutions.

For the case of the A_2 spin chain we expect that the six polynomial solutions (29) play the same role. Indeed, let us recall that the polynomials Q_1, P_1 and R_1 are the solutions of (15):

$$\begin{pmatrix} Q_1(\lambda - \frac{3\mathrm{i}}{2}) & Q_1(\lambda - \frac{\mathrm{i}}{2}) & Q_1(\lambda + \frac{\mathrm{i}}{2}) & Q_1(\lambda + \frac{3\mathrm{i}}{2}) \\ P_1(\lambda - \frac{3\mathrm{i}}{2}) & P_1(\lambda - \frac{\mathrm{i}}{2}) & P_1(\lambda + \frac{\mathrm{i}}{2}) & P_1(\lambda + \frac{3\mathrm{i}}{2}) \\ R_1(\lambda - \frac{3\mathrm{i}}{2}) & R_1(\lambda - \frac{\mathrm{i}}{2}) & R_1(\lambda + \frac{\mathrm{i}}{2}) & R_1(\lambda + \frac{3\mathrm{i}}{2}) \end{pmatrix} \begin{pmatrix} (\lambda + \frac{\mathrm{i}}{2})^N \\ -T^{+}(\lambda) \\ T^{-}(\lambda) \\ -(\lambda - \frac{\mathrm{i}}{2})^N \end{pmatrix} = 0. \tag{30}$$

Table 1.

Number	$Q(\lambda)$	$P(\lambda)$	$R(\lambda)$
1	1	λ	$\lambda^5 + \frac{5}{3}\lambda^3$
2	1	$\lambda^2 + \frac{\lambda}{\sqrt{3}}$	$\lambda^4 - \frac{2\lambda^3}{\sqrt{3}} - \sqrt{3}\lambda$
3	1	$\lambda^2 - \frac{\lambda}{\sqrt{3}}$	$\lambda^4 + \frac{2\lambda^3}{\sqrt{3}} + \sqrt{3}\lambda$
4	λ	$\lambda^2 + \frac{1}{3}$	λ^3

Excluding $T^{\pm}(\lambda)$ from this system we obtain the following equation:

$$\begin{vmatrix} Q_1(\lambda - \mathrm{i}) & Q_1(\lambda) & Q_1(\lambda + \mathrm{i}) \\ P_1(\lambda - \mathrm{i}) & P_1(\lambda) & P_1(\lambda + \mathrm{i}) \\ R_1(\lambda - \mathrm{i}) & R_1(\lambda) & R_1(\lambda + \mathrm{i}) \end{vmatrix} = \lambda^N. \tag{31}$$

This equation is the analogue of the fundamental ‘Wronskian’ (16) from [1]. As in the case of A_1, (31) can be considered as the starting point for the construction of polynomials Q_1, P_1, R_1 and consequently the transfer matrices $T^{\pm}(\lambda)$:

$$\begin{vmatrix} Q_1(\lambda - 3\mathrm{i}/2) & Q_1(\lambda \pm \mathrm{i}/2) & Q_1(\lambda + 3\mathrm{i}/2) \\ P_1(\lambda - 3\mathrm{i}/2) & P_1(\lambda \pm \mathrm{i}/2) & P_1(\lambda + 3\mathrm{i}/2) \\ R_1(\lambda - 3\mathrm{i}/2) & R_1(\lambda \pm \mathrm{i}/2) & R_1(\lambda + 3\mathrm{i}/2) \end{vmatrix} = T^{\pm}(\lambda). \tag{32}$$

Consider, for example, the case of a three-site chain, i.e. $N = 3$. The full set of polynomial solutions of (31) in this case is shown in table 1.

The four solutions given in table 1 correspond to four irreducible representations which enter the decomposition of the product of $N = 3$ fundamental representations

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10}. \tag{33}$$

Note that we can express in terms of these polynomials not only the $T^{\pm}(\lambda)$ which correspond to the fundamental representation in auxiliary space, but also the transfer matrices for any other representation of A_2.

Similar relations exist also for the polynomials Q_2, P_2, R_2. Taking into account the first equation of (20) one can establish that these relations are not independent.

In [6] the A_n case of nested Bethe ansatz equations was considered using analogues of Baxter’s *TQ* equations. However, in their approach the ‘regularization’ by means of an ‘external magnetic field’ is essential and it is not known how to remove this regularization. At present we are therefore unable to compare our results.

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